Enhancing the Reasoning-and-Proving Content of Textbook Tasks: A Site for Teacher Professional Development

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This work a is part of the Cases of Reasoning and Proving in Secondary Mathematics (CORP) Project, National Science Foundation Discovery Research K-12 program (Pis Margaret S. Smith & Fran Arbaugh). The views expressed here do not necessarily reflect the views of the National Science Foundation.
Session Agenda

- Perspectives on Reasoning-and-Proving
- Analyzing Some Reasoning-and-Proving Tasks
- Discussion of Task Modification Strategies
- Engaging in a Task Modification Activity
The work in which mathematicians engage that culminates in a formal proof involves searching a mathematical phenomena for patterns, making conjectures about those patterns, and providing informal arguments demonstrating the viability of the conjectures.

Lakatos, 1976
In the Common Core...
Practice 3: Construct viable arguments and critique the reasoning of others.
Practice 7: Look for and make use of structure
Practice 8: Look for and express regularity in repeated reasoning

Common Core Standards, 2010
By focusing primarily on the final product—that is, the proof—students are not afforded the same level of scaffolding used by professional users of mathematics to establish mathematical truth. Therefore, reasoning-and-proving should be defined to encompass the breadth of activity associated with:

- identifying patterns
- making conjectures
- providing proofs, and
- providing non-proof arguments.

Stylianides, 2008
Analysis of Connected Mathematics
- ~40% of tasks involve reasoning-and-proving
- However, only 3% of tasks involved conjecturing
- Almost all proving tasks were demonstrations
  (Stylianides, 2005)

Analysis of 4 Algebra I, II, and Precalculus texts:
- 5.5% of problems required reasoning
- Less than 1% provided conjecturing opportunities
  (Johnson, Thompson, & Senk, 2010)
Comparing two versions of the same task
Comparing Two Versions of a Task

- Compare each task to its adapted version (A to A’, B to B’, C to C’)
- Determine how each pair of task is the same and how it is different
- Look across the three sets and consider:
  - what goals might the task modifier have had in mind for modifying the tasks?
  - whether the differences between a task and its adaptation matter?
Comparing Two Versions of a Task

**TASK A**

MAKING CONJECTURES Complete the conjecture based on the pattern you observe in the specific cases.

**29. Conjecture:** The sum of any two odd numbers is _____?

- $1 + 1 = 2$
- $7 + 11 = 18$
- $1 + 3 = 4$
- $13 + 19 = 32$
- $3 + 5 = 8$
- $201 + 305 = 506$

**30. Conjecture:** The product of any two odd numbers is ____?

- $1 \times 1 = 1$
- $7 \times 11 = 77$
- $1 \times 3 = 3$
- $13 \times 19 = 247$
- $3 \times 5 = 15$
- $201 \times 305 = 61,305$

**TASK A’**

For problems 29 and 30, complete the conjecture based on the pattern you observe in the examples. Then explain why the conjecture is always true or show a case in which it is not true.

MAKING CONJECTURES Complete the conjecture based on the pattern you observe in the specific cases.

**29. Conjecture:** The sum of any two odd numbers is _____?

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Comparing Tasks A and A’

**Similar**
- Both ask students to complete a conjecture about odd numbers based on a set of finite examples that are provided.

**Different**
- Task A’ asks students to develop an argument that explains why the conjecture is always true (or not).
- Task A can be completed with limited effort; Task A’ requires considerable effort – students need to figure out why this conjecture holds up.
Comparing Two Versions of a Task

TASK B

VISUAL REASONING Explain why the following method of drawing a parallelogram works. State a theorem to support your answer.

1. Use a ruler to draw a segment and its midpoint.
2. Draw another segment so the midpoints coincide.
3. Connect the endpoints of the segments.

TASK B’

Consider the construction below.

Use the construction with a variety of starting segments.

Make a conjecture about the type of figure that the construction produces.

Using the properties that you know about that figure, make a mathematical argument that explains why that figure is produced each time by the construction.

1. Use a ruler to draw a segment and its midpoint.
2. Draw another segment so the midpoints coincide.
3. Connect the endpoints of the segments.
Comparing Tasks B and B’

Similar
- Same geometry content
- Based on the same construction
- Relates to a specific theorem about the diagonals of a parallelogram

Different
- Task B: perform the construction once, then explain why the figure is a parallelogram
- Task B’: perform the construction several times, then conjecture about the type of figure produced
- Task B: state a theorem
- Task B’: make an argument using known properties to explain why the same figure is produced each time
Comparing Two Versions of a Task

TASK C

GEOMETRY For Exercises 45 and 46, use the diagram below that shows the perimeter of the pattern consisting of trapezoids.

1. Find the perimeter of the first four trapezoids shown above.
2. Find the perimeter of the pattern containing 12 trapezoids without drawing a picture.
3. Write a generalization that can be used to find the perimeter of a pattern containing \( n \) trapezoids.
4. Use the diagram to explain why your generalization always works.

12 April 2011

NCSM Annual Meeting, Indianapolis IN

45. Write a formula that can be used to find the perimeter of a pattern containing \( n \) trapezoids.

46. What is the perimeter of the pattern containing 12 trapezoids?

TASK C’

All trapezoids follow this pattern

1. Find the perimeter of the first four trapezoids shown above.
2. Find the perimeter of the pattern containing 12 trapezoids without drawing a picture.
3. Write a generalization that can be used to find the perimeter of a pattern containing any number of trapezoids.
4. Use the diagram to explain why your generalization always works.
Comparing Tasks C and C’

Similar

- Both tasks ask students to find the perimeter of the pattern containing 12 trapezoids.
- Both tasks ask students to find a rule that can be used to find the perimeters of a pattern containing any number of trapezoids.
- Both tasks require students to identify the pattern of growth of the perimeter.

Different

- Task C’ provides more scaffolding by first asking students to find the perimeter of the first four patterns.
- Task C: finding the perimeter of the 12 trapezoid pattern comes after finding the general rule.
- Task C’: finding the perimeter of the 12 trapezoid pattern comes before finding the general rule.
- Task C’ asks students to create a generalization and use the diagram to explain why it always works.
Goals for modifying tasks

- Provide more and different opportunities to reason and prove than the original tasks
- Create low threshold-high ceiling tasks that give multiple points of entry to students
- Engage students in investigation and conjecture instead of just giving answers
- Generate proof (or proof-like) arguments (without needing to use the term proof)
TASK A’

For problems 29 and 30, complete the conjecture based on the pattern you observe in the examples. Then explain why the conjecture is always true or show a case in which it is not true.

MAKING CONJECTURES Complete the conjecture based on the pattern you observe in the specific cases.

29. **Conjecture:** The sum of any two odd numbers is _____?
   
   - $1 + 1 = 2$
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Comparing Two Versions of a Task

TASK B’

Consider the construction below.

Use the construction with a variety of starting segments.

Make a conjecture about the type of figure that the construction produces.

Using the properties that you know about that figure, make a mathematical argument that explains why that figure is produced each time by the construction.

1. Use a ruler to draw a segment and its midpoint.
2. Draw another segment so the midpoints coincide.
3. Connect the endpoints of the segments.
1. Find the perimeter of the first four trapezoids shown above.

2. Find the perimeter of the pattern containing 12 trapezoids without drawing a picture.

3. Write a generalization that can be used to find the perimeter of a pattern containing any number of trapezoids.

4. Use the diagram to explain why your generalization always works.
The power of modification strategies

- Target the practice of reasoning-and-proving
- Can apply across content and grade levels
- Empower teachers to use existing resources as compared to finding or developing tasks on their own
Engaging teachers in task modification

- Develop a list of strategies with teachers
- Have teams of teachers modify the same task
  - ...and teach it if possible, comparing the results
- Encourage teachers to solve original and modified tasks
  - ...often reveals underlying assumptions or implied procedures that can limit student thinking
With your small group, modify one of the three tasks (D, E, F) on your blue sheet

Describe how your modifications support students in engaging in reasoning-and-proving

- Provide more and different opportunities to reason and prove than the original tasks
- Create low threshold-high ceiling tasks that give multiple points of entry to students
- Engage students in investigation and conjecture instead of just giving answers
- Generate proof arguments (without needing to use the term proof)
Task D

**DEVELOPING PROOF** In Exercises 20–23, use the following information.

Dan is trying to figure out how to prove that $\angle 5 \cong \angle 6$ below. First he wrote everything that he knew about the diagram, as shown below in blue.

Given: $m \perp n$, $\angle 3$ and $\angle 4$ are complementary.

Prove: $\angle 5 \cong \angle 6$

$m \perp n \rightarrow \angle 3$ and $\angle 6$ are complementary.

$\angle 4$ and $\angle 5$ are vertical angles. $\rightarrow \angle 4 \cong \angle 5$

20. Write a justification for each statement Dan wrote in blue.

21. After writing all he knew, Dan wrote what he was supposed to prove in red. He also wrote $\angle 4 \cong \angle 6$ because he knew that if $\angle 4 \cong \angle 6$ and $\angle 4 \cong \angle 5$, then $\angle 5 \cong \angle 6$. Write a justification for this step.

22. How can you use Dan’s blue statements to prove that $\angle 4 \cong \angle 6$?

23. Copy and complete Dan’s flow proof.

(McDougal Littell (2004), Geometry, p. 140)
Task E

Use the following procedure to graph the linear functions below.

a. Write the function in standard form.  
   \(2x + 5y = 30\)

b. Put your hand over the \(x\) term and solve for \(y\). Plot this point on your \(y\)-axis.  
   \(2x + 5y = 30\)
   \(y = 6\)

c. Put your hand over the \(y\) term and solve for \(x\). Plot this point on your \(x\)-axis.  
   \(2x + 5y = 30\)
   \(x = 15\)

d. Draw a line connecting your points.

1. \(4x + 9y = 36\)  
   2. \(x + 7y = 7\)  
   3. \(-5x + y = 20\)  
   4. \(3x - 2y = 42\)  
   5. \(4x + 3y = 18\)  
   6. \(-2x - 6y = 24\)  
   7. \(5x - 3y = 16\)  
   8. \(-4x + y = 10\)
Task F

Jamaal is making a triangular wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?
More information on CORP

- Curriculum under revision following pilot
- Field testing opportunities available
  - Contact:
    - Peg Smith pegs@pitt.edu
    - Fran Arbaugh arbaugh@psu.edu
- Anticipated public availability in 2013
## Comparing tasks and their modifications

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<thead>
<tr>
<th>Making Mathematical Generalizations</th>
<th>Providing Support to Mathematical Claims</th>
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</thead>
<tbody>
<tr>
<td><strong>Identifying a pattern</strong></td>
<td><strong>Making a conjecture</strong></td>
</tr>
<tr>
<td><strong>Providing a proof</strong></td>
<td><strong>Providing a non-proof argument</strong></td>
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<tr>
<th><strong>B'</strong></th>
<th><strong>A, A'</strong></th>
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<td><strong>C, C'</strong></td>
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<td><strong>B'</strong></td>
<td><strong>C'</strong></td>
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Task modification strategies

1) Adding the requirement that students make observations before they engage in proving activities.

2) Adding the requirement that students explore or identify a pattern from given data or generate examples in order to search for patterns.

3) Adding the requirement that students make or revise conjectures.

4) Adding the requirement that students provide a mathematical argument or proof.

5) Removing or reducing scaffolding from a proof-related task so that the task is open to different approaches and does not tell exactly what to do or how to do it.

6) Adding the requirement that students write a proof using a different representation(s) (flow chart, 2-column, paragraph) that the way the proof was initially presented, and/or make comparisons between different proofs.